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LETTER TO THE EDITOR

Remarks on the local time rescaling in path integration

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Abstract. The local time transformation used in path integration is shown to be equivalent to standard procedures applied to Schrödinger's equation.

Since Duru and Kleinert [1] introduced the idea of local time rescaling in path integration this method has been applied very successfully in recent years. By reparametrizing the paths with a new, in general, path dependent time many problems have become solvable by path integration. However, there exists no rigorous proof for the validity of this procedure. The local time rescaling technique should be understood as a recipe. For a detailed discussion we refer to the work of Inomata [2–4].

In this letter we want to show that the recipe mentioned above is equivalent to standard techniques for the solution of Schrödinger's equation. First we present the general treatment which transforms the stationary Schrödinger equation for a given potential $V(x)$ into one for a potential $\tilde{V}(z)$ with fixed energy $\tilde{E} = \tilde{V}_0$, whose solutions $\varphi_n(z)$ and $\tilde{E}_n$ are assumed to be known. The energy eigenfunctions $\Phi_n(x)$ and energy eigenvalues E_n of the original problem are then expressed in terms of $\varphi_n(z)$ and $\tilde{E}_n$. Secondly, we present the time transformation technique of path integration and show its similarity to the methods used for Schrödinger's equation.

Let us consider the stationary Schrödinger equation for a one-dimensional particle with mass M in the potential $V(x)$

$$\left(\frac{d^2}{dx^2} - \frac{2M}{\hbar^2} V(x) + \frac{2M}{\hbar^2} E\right)\Phi(x) = 0. \tag{1}$$

If the solution of (1) is not known by standard techniques like factorization, algebraization or direct identification with hypergeometric or confluent hypergeometric function one usually tries to transform (1) into another Schrödinger equation whose solution is know by one of the above methods [5, 6]. Hence, the general ansatz we can start with is

$$x = f(z) \qquad \Phi(x) = g(z)\varphi(z) \tag{2}$$

which should satisfy a differential equation similar to (1)

$$\left(\frac{d^2}{dz^2} - \frac{2M}{\hbar^2} \tilde{V}(z) + \frac{2M}{\hbar^2} \tilde{E}\right)\varphi(x) = 0. \tag{3}$$

We assume that the solution of this problem is known and given by $\varphi_n(z)$ and $\tilde{E}_n$. For simplicity we assume a discrete spectrum only. Using standard calculus one obtains

$$\frac{d^2}{dx^2}\Phi(x) = \frac{1}{f'}\left[\frac{g}{f'}\varphi'' + \frac{2f'g' - gf''}{f'^2}\varphi' + \left(\frac{g''}{f'} - \frac{f''g'}{f'^2}\right)\varphi\right] \tag{4}$$

where $f' \equiv f'(z) = \mathrm{d}f(z)/\mathrm{d}z$ and similar for $g(z)$ and $\varphi(z)$. Obviously the vanishing of the term linear in $\varphi'(z)$ implies the condition $2f'(z)g'(z) = g(z)f''(z)$ which leads to

$$g(z) = c\sqrt{f'(z)}. \tag{5}$$

In the above c is a constant of integration. Inserting this result in (1) and comparing it with (3) gives

$$\tilde{V}(z) = f'^2(z)[V(f(z)) - E] - \frac{\hbar^2}{4M}\left(\frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right) + \tilde{V}_0 \tag{6}$$

$$\tilde{E}(E) = \tilde{V}_0. \tag{7}$$

Here $\tilde{V}_0$ is an arbitrary constant and may be chosen to zero without loss of generality. Hence, the solution of the original problem (1) may be expressed in terms of the solution of (3)

$$\Phi_n(x) = c_n\sqrt{f'(z)}\,\varphi_n(z) \tag{8}$$

$$\{E_n\} = \{E|\tilde{E}_n(E) = \tilde{V}_0\}. \tag{9}$$

The constants c_n may be obtained by normalization and $z = f^{-1}(x)$. It is interesting to note that the additional potential appearing in (6) due to the kinetic term (4) is proportional to the Schwarz derivative of $f(z)$

$$\mathscr{S}f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

Now we will show that the local time rescaling in path integration is equivalent to the above procedure for Schrödinger's equation. The aim of time transformation is to change the path integral for a non-integrable problem into one whose path integral solution is known. These are the harmonic oscillator with additional inverse square potential [7], the Pöschl–Teller and modified Pöschl–Teller potential [8]. The starting point is the functional integral for the promotor [2–4]

$$P(x'', x'; \tau) = \lim_{N\to\infty} \int \prod_{j=1}^{N} \mathrm{e}^{(\mathrm{i}/\hbar)W_j} \prod_{j=1}^{N} \left(\frac{M}{2\pi\mathrm{i}\hbar\tau_j}\right)^{1/2} \prod_{j=1}^{N-1} \mathrm{d}x_j \tag{10}$$

where

$$W_j = \frac{M}{2\tau_j}(\Delta x_j)^2 - V(x_j)\tau_j + E\tau_j \tag{11}$$

is Hamilton's characteristic function and $\tau = \Sigma_{j=1}^{N} \tau_j$, $\Delta x_j = x_j - x_{j-1}$, $x'' = x_N$ and $x' = x_0$. The promotor itself has no physical interpretation. However, the energy dependent Green function is obtained by integration:

$$G(x'', x'; E) = \frac{1}{\mathrm{i}\hbar}\int_0^\infty P(x'', x'; \tau)\,\mathrm{d}\tau. \tag{12}$$

The nonlinear transformation $x = f(z)$, which is the same as in (2), is accompanied by a local rescaling of the time intervals [2–4]

$$\tau_j = h(z_j)h(z_{j-1})\sigma_j \tag{13}$$

with the global scaling property

$$\tau = h(z'')h(z')\sigma \qquad \sigma = \sum_{j=1}^{N} \sigma_j. \tag{14}$$

The requirement that the kinetic term $(\Delta x_j)^2/\tau_j$ be changed into a kinetic term in z-space and σ-time, namely $(\Delta z_j)^2/\sigma_j$, leads to the condition

$$h(z)=f'(z). \tag{15}$$

Note that in path integration $(\Delta z_j)^2=\mathcal{O}(\sigma_j)$ and therefore terms up to $\mathcal{O}((\Delta z_j)^4)$ have to be considered in the expansion of the kinetic term in (11). A detailed calculation gives

$$P(x'',x';\tau)=[f'(z')f'(z'')]^{-1/2}\tilde{K}(z'',z';\sigma)\exp\{(\mathrm{i}/\hbar)\tilde{V}_0\sigma\} \tag{16}$$

where

$$\tilde{K}(z'',z';\sigma)=\lim_{N\to\infty}\int\prod_{j=1}^{N}\mathrm{e}^{(\mathrm{i}/\hbar)\tilde{S}_j}\prod_{j=1}^{N}\left(\frac{M}{2\pi\mathrm{i}\hbar\sigma_j}\right)^{1/2}\prod_{j=1}^{N-1}\mathrm{d}z_j \tag{17}$$

is a path integral for the propagator of a particle evolving with time σ in the potential $\tilde{V}(z)$ given in (6). The corresponding short time action reads

$$\tilde{S}_j=\frac{M}{2\sigma_j}(\Delta z_j)^2-\tilde{V}(z_j)\sigma_j. \tag{18}$$

With the help of (12) and (14) we express the Green function as

$$G(x'',x';E)=\frac{1}{\mathrm{i}\hbar}[f'(z')f'(z'')]^{1/2}\int_0^\infty\tilde{K}(z'',z';\sigma)\,\mathrm{e}^{(\mathrm{i}/\hbar)\tilde{V}_0\sigma}\,\mathrm{d}\sigma. \tag{19}$$

After path integration the unphysical propagator (17) may be written as

$$\tilde{K}(z'',z';\sigma)=\sum_n\exp\{-(\mathrm{i}/\hbar)\tilde{E}_n\sigma\}\varphi_n(z'')\varphi_n^*(z'). \tag{20}$$

Again we have assumed a discrete spectrum only.

Now the Green function can be calculated and yields

$$G(x'',x';E)=[f'(z')f'(z'')]^{1/2}\sum_n\frac{\varphi_n(z'')\varphi_n^*(z')}{\tilde{E}_n(E)-\tilde{V}_0}. \tag{21}$$

A comparison with the standard form

$$G(x'',x';E)=\sum_n\frac{\Phi_n(x'')\Phi_n^*(x')}{E-E_n} \tag{22}$$

leads to the wavefunctions (8) with $c_n=[(\partial\tilde{E}(E)/\partial E)|_{E=E_n}]^{-1/2}$ and the energy spectrum is that given in (9).

In this letter we have shown that the local time transformation technique is equivalent to standard procedures in solving the Schrödinger equation. The recipe for nonlinear space transformation accompanied by a rescaling of time slices in path integration gives the same result as substitution of dependent and independent variable in Schrödinger's equation.

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